Maths for Computing Tutorial 6

1. Let $X = \{x \mid x \text{ is a real number such that } 1 < x < 2\}$. Then prove that |X| = |R|.

2. Prove that $|P(\mathbb{Z})| = |\mathbb{R}|$, where $P(\mathbb{Z})$ is the power set of \mathbb{Z} .

3. Prove that there can be no bijection between \mathbb{Z}^+ and $P(\mathbb{Z}^+)$.

4. Prove that set of functions from \mathbb{Z}^+ to \mathbb{Z}^+ is uncountable.

5. Give a bijection from (0,1] to (0,1).

6. Prove that a finite poset will always have a maximal element.

7. A busy airport sees 1500 takeoffs per day. Prove that there are two planes that must take off within a minute of each other.

8. The set M consists of 9 positive integers, none of which has a prime divisor larger than 6. Prove that M has two elements whose product is the square of an integer.

9. Let *H* be a 10-element set of 2-digit positive integers. Prove that *H* has two disjoint subsets *A* and *B* so that the sum of the elements of *A* is equal to the sum of the elements of *B*.

10. Suppose 51 girls and 51 boys are seated around a circular table. Is it true that there is always a person both of whose neighbours are boys. Prove it formally or disprove it by giving a counter-example.

11. Suppose we are given a sequence of *n* integers $a_1, a_2, ..., a_n$, which need not be distinct. Prove that there is always a subsequence of consecutive numbers $a_k, a_{k+1}, ..., a_l$ in the given sequence,

whose sum $\sum_{i=k}^{l} a_i$ is a multiple of *n*.

12. Let $n \ge 1$. Prove that if we select any n + 1 integers from $\{1, 2, ..., 2n\}$, then there exists two integers, say a and b, such that a % b = 0.

Tutorial 6

Solution 1

We can prove |X| = |R| using Schröder-Bernstein theorem. That is, we will give an injection from *X* to *R* and from *R* to *X*.

Injection from *X* **to** *R*: $f : X \to R$ is f(x) = x. Clearly, *f* is an injection.

Injection from *R* to *X*: $f : R \to X$ is $f(x) = \frac{2^x}{1+2^x} + 1$.

Range of f is X: Since 2^x is positive for any $x \in R$, $\frac{2^x}{1+2^x} > 0$. Therefore, $\frac{2^x}{1+2^x} + 1 > 1$. Also, $2^x < 1 + 2^x \implies \frac{2^x}{1+2^x} < 1 \implies \frac{2^x}{1+2^x} + 1 < 2$. Hence, f(x) < 2.

f is one-to-one: Let $x, y \in R$ such that f(x) = f(y). Then,

2^{x} , 2^{y} , 1		
	$\frac{1}{1+2^x} + 1 = \frac{1}{1+2^y} + \frac{1}{1+2^y$	- 1
\Rightarrow	$\frac{2^x}{1+2^x} = \frac{2^y}{1+2^y}$	
\Rightarrow	$\frac{1+2^x}{2^x} = \frac{1+2^y}{2^y}$	
\Rightarrow	$1 + \frac{1}{2^x} = 1 + \frac{1}{2^y}$	
\Rightarrow	$\frac{1}{2^x} = \frac{1}{2^y}$	
\Rightarrow	$2^x = 2^y$	
\Rightarrow	x = y	(take log on both sides.).

Solution 2

First note that $|P(\mathbb{Z})| = |P(\mathbb{Z}^+)|$ (using the fact that there is a bijection between \mathbb{Z} and \mathbb{Z}^+) and $|\mathbb{R}^+| = |\mathbb{R}|$ (using Schröder-Bernstein). Therefore, it is enough to prove $|P(\mathbb{Z}^+)| = |\mathbb{R}^+|$, which we will do using Schröder-Bernstein.

Injection from $P(\mathbb{Z}^+)$ to \mathbb{R}^+ : Let f be a function from $P(\mathbb{Z}^+)$ to \mathbb{R}^+ such that when $x = \emptyset$, f(x) = 1 and when $x \neq \emptyset$, $f(x) = .b_1b_2b_3...$, where $b_i = 1$ if $i \in x$, else $b_i = 0$. Clearly, f is an injection.

Injection from \mathbb{R}^+ *to* $P(\mathbb{Z}^+)$: Again $P(\mathbb{Q}^+)$ has a bijection to $P(\mathbb{Z}^+)$ and there in an injection from \mathbb{R}^+ to (0,1). So it is enough to give an injection from (0,1) to $P(\mathbb{Q}^+)$.

Let *f* be a function from (0,1) to $P(\mathbb{Q}^+)$ such that $f(x) = \{q \le x \mid q \in \mathbb{Q}^+\}$. Range of *f* is clearly $P(\mathbb{Q}^+)$.

We will now prove that f is one-to-one. Let x_1 and x_2 be two distinct real numbers. WLOG assume that $x_1 < x_2$. If x_2 is rational, then $x_2 \in f(x_2)$, but $x_2 \notin f(x_1)$. Hence, $f(x_1) \neq f(x_2)$. Now, assume that x_2 is irrational. Let $x_1 = .a_1a_2a_3...$ where $a_i \in \{0,1,...,9\}$ and $x_2 = .b_1b_2b_3...$, where $b_i \in \{0,1,...,9\}$. Since $x_1 < x_2$, then there must exists an i such that $a_i \neq b_i$. Let k be the first integer where $a_k \neq b_k$. Clearly, $a_k < b_k$. Now consider the number $y = .b_1b_2b_3...b_k$. y is clearly a rational number, which is greater than x_1 and lesser than x_2 (as x_2 must have non-zero digits after b_k). Therefore, $y \in f(x_2)$, but $y \notin f(x_1)$. Hence, $f(x_1) \neq f(x_2)$.

Solution 3

We first give a bijection, say f, from $P(\mathbb{Z}^+)$ to the set of infinite length binary strings, say S. Let s be a subset of \mathbb{Z}^+ , then $f(s) = b_1 b_2 \dots b_k \dots$, where $b_i = 1$ if $i \in s$, else $b_i = 0$. f is clearly a bijection. We will prove now that there cannot be a bijection between \mathbb{Z}^+ and S. This is sufficient because if there is a bijection between \mathbb{Z}^+ and $P(\mathbb{Z}^+)$ and a bijection between $P(\mathbb{Z}^+)$ and S, then there will also be a bijection between \mathbb{Z}^+ and S.

Suppose there is a bijection between \mathbb{Z}^+ and S. Then elements of S can be listed out as s_1, s_2, s_3, \ldots . Now we create a new infinite length binary string w in the following manner. The *i*th bit of w, that is, $w_i = 1 - s_{ii}$, where s_{ii} is the *i*th bit of s_i . Now s cannot be present in the sequence s_1, s_2, s_3, \ldots as it differs from every s_k string on the *k*th bit.

Solution 4

Suppose the set of functions from \mathbb{Z}^+ to \mathbb{Z}^+ is finite. Then, there will be some k many different functions from \mathbb{Z}^+ to \mathbb{Z}^+ , say f_1, f_2, \ldots, f_k for some positive integer k. Let f be a function defined as $f(i) = max(f_1(i), f_2(i), \ldots, f_k(i)) + 1$. Clearly, f is different from all the f_i s. Hence, a contradiction.

Now suppose the set of functions from \mathbb{Z}^+ to \mathbb{Z}^+ is countably infinite. Then there will be a sequence *F* of functions f_1, f_2, f_3, \ldots in which every function from \mathbb{Z}^+ to \mathbb{Z}^+ is present exactly once.

Now, construct a function f from \mathbb{Z}^+ to \mathbb{Z}^+ so that

$$f(i) = 1$$
, if $f_i(i) \neq 1$
 $f(i) = 2$, if $f_i(i) = 1$

This newly constructed function f is not in the sequence F as f differs from every f_k in F as $f(k) \neq f_k(k)$. Hence, a contradiction.

Solution 5

There is clearly a bijection from $\{1,2,3,...\}$ to $\{2,3,4,...\}$. By taking reciprocal of each element we can have a bijection, say *g*, from $\{1,1/2,1/3,...\}$ to $\{1/2,1/3,1/4,...\}$.

Let $A = \{1, 1/2, 1/3, ...\}$ and $B = (0, 1] \setminus A$. We can now give a bijection, say f, from (0, 1] to (0, 1). If $x \in A$, then f(x) = g(x). Else, f(x) = x. We will prove now that f is a bijection.

f is one-to-one: Let x_1 and x_2 be two distinct elements. If both x_1 and x_2 are in A, $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$. Since *g* is a bijection, $g(x_1) \neq g(x_2)$. If both x_1 and x_2 are in B, $f(x_1) \neq f(x_2)$ as $f(x_1) = x_1$ and $f(x_2) = x_2$. Now WLOG assume that $x_1 \in A$ and $x_2 \in B$. Then $f(x_1) \notin B$, but $f(x_2) \in B$. Hence, again $f(x_1) \neq f(x_2)$.

f is onto: Let *y* be an element in (0,1). If *y* is of the form 1/w, then $f(g^{-1}(y)) = y$. Otherwise, f(y) = y.

Solution 6

We will prove it using induction.

Basis Step: Let (X, R) be a poset, where |X| = 1. Then clearly the only element will be maximal element.

Inductive Step: Let us assume that every poset of size less than *n* has a maximal element. Let (X, R) be a poset, where |X| = n. Pick any element *x* of *X*. If *x* is a maximal element, we are done. If not, consider the set $S = \{y \mid x \prec y\}$. Since *x* is not a maximal element *S* will be a non-empty set. Now, consider the poset (S, R). From inductive hypothesis, (S, R) has a maximal element, say *w*. We can prove now that *w* is also a maximal element of (X, R). Suppose not, then there must be some element of *X* say *z* such that $w \prec z$. But $x \prec w$. This implies that $x \prec z$. Hence, *z* should be a member of *S* and *w* cannot be a maximal element of (S, R). This is a contradiction.

Solution 7

There are 1440 minutes in one day. Since flights are 1500, from pigeonhole principle we can say that there must be at least two flights which take off within a minute.

Solution 8

Every number of M can be written as $2^a 3^b 5^c$ as they do not have any factor more than 6. Let us create 8 pigeonholes each corresponding to three parities, i.e., (odd, odd, odd), (odd, odd, even), (odd, even, odd), (odd, even, even), etc. Now we assign every number from M to one of the pigeonholes if the parity of powers of 2,3 and 5 in the factorisation of that number matches with parities of pigeonholes. From pigeonhole principle, two numbers, say $2^{a_1}3^{b_1}5^{c_1}$ and $2^{a_2}3^{b_2}5^{c_2}$, will get the same pigeonhole. That means a_1 and a_2 are of same parity, b_1 and b_2 are of same parity, and

 c_1 and c_2 are of same parity. This implies $(a_1 + a_2)/2$, $(b_1 + b_2)/2$, and $(c_1 + c_2)/2$, are integers. Hence, product of these numbers, i.e., $2^{a_1+a_2}.3^{b_1+b_2}.5^{c_1+c_2}$ is the square of the integer $2^{(a_1+a_2)/2}.3^{(b_1+b_2)/2}.5^{(c_1+c_2)/2}$.

Solution 9

Every subset of *H* can have the sum ranging from 0 (for subset Ø) to 945 (take the subset as $\{90,91,\ldots,99\}$). The total number of subsets of $H = 2^{10} = 1024$. Taking subsets as pigeons and sums as pigeonholes, we can say that at least two subsets, say *A* and *B*, of *H* will have the same sum. If *A* and *B* are disjoint, we are done. Else, we can drop the common elements of *A* and *B*, i.e., $A \setminus (A \cap B)$ and $B \setminus (A \cap B)$ will be two disjoint subsets of *H* with the same sum.

Solution 10

Consider any seating of 51 girls and 51 boys around a circular table. Let's call a set of boys, say X, sitting together a boy group if there is no proper superset of boys, say Y, of X such that Y are also sitting together. Similarly, we define girl groups. It is easy to see that number of boy groups is equal to the number of girl groups as they are sitting around a circular table. (Without loss of generality, let's pick any girl group as the first group of sequence S and start adding other groups (of boy's and girl's) in a clockwise order to S. Suppose the $S = \{s_1, s_2, s_3, ..., s_k\}$, where s_1 is a girl group, s_2 is a boy group, and so on. s_k in S has to be a boy group, otherwise s_k and s_1 will together form a girl group. This proves that k is even and the number of boy groups is equal to the number of girl groups.)

If there is boy group of size 3 or more, then we are done as some boy will have two boys his neighbours. If there is a girl group of size 1, then again we are done as that girl will have two boys her neighbours.

Now suppose both these conditions are false. That is, every boy group is of size at most 2 and every girl group is of size at least 2. If all boy groups are of size at most 2, then there will be at least 26 boy groups. If every girl group is of size at least 2, then there will be at most 25 girl groups. But this is a contradiction as number of girl groups is the same as the number of boy groups.

Solution 11

Take n subsequences of consecutive numbers:

$$A_{1} = a_{1}$$

$$A_{2} = a_{1} + a_{2}$$

$$A_{3} = a_{1} + a_{2} + a_{3}$$
.....
$$A_{n} = a_{1} + a_{2} + \dots + a_{n}$$

Now take modulo of A_i s with n. If there is an i such that $A_i \% n = 0$, then we are done. Otherwise, from pigeonhole principle there will be i and j, where i < j, such that $A_i \% n = A_j \% n$. That implies, $(A_j - A_i) \% n = 0$. Clearly, $A_j - A_i$ is a subsequence of consecutive integers $a_{i+1}, a_{i+2}, ..., a_j$.

Solution 12

Let *x* be any number in [2*n*]. Then *x* can be written in the form of $2^k y$, where $k \ge 0$ and *y* is an odd number. Now, we can pick any n + 1 numbers from [2*n*] and these will be our pigeons. Create *n* pigeonholes corresponding to each odd number in [2*n*]. Put a pigeon, say $x = 2^k y$, to the pigeonhole *y*. From pigeonhole principle, we can say that there will be at least one pigeonhole, say *y*, that contains two pigeons, say $a = 2^{k_1}y$ and $b = 2^{k_2}y$, such that $k_1 > k_2$. Clearly, *a* is divisible by *b*. Hence, a % b = 0.